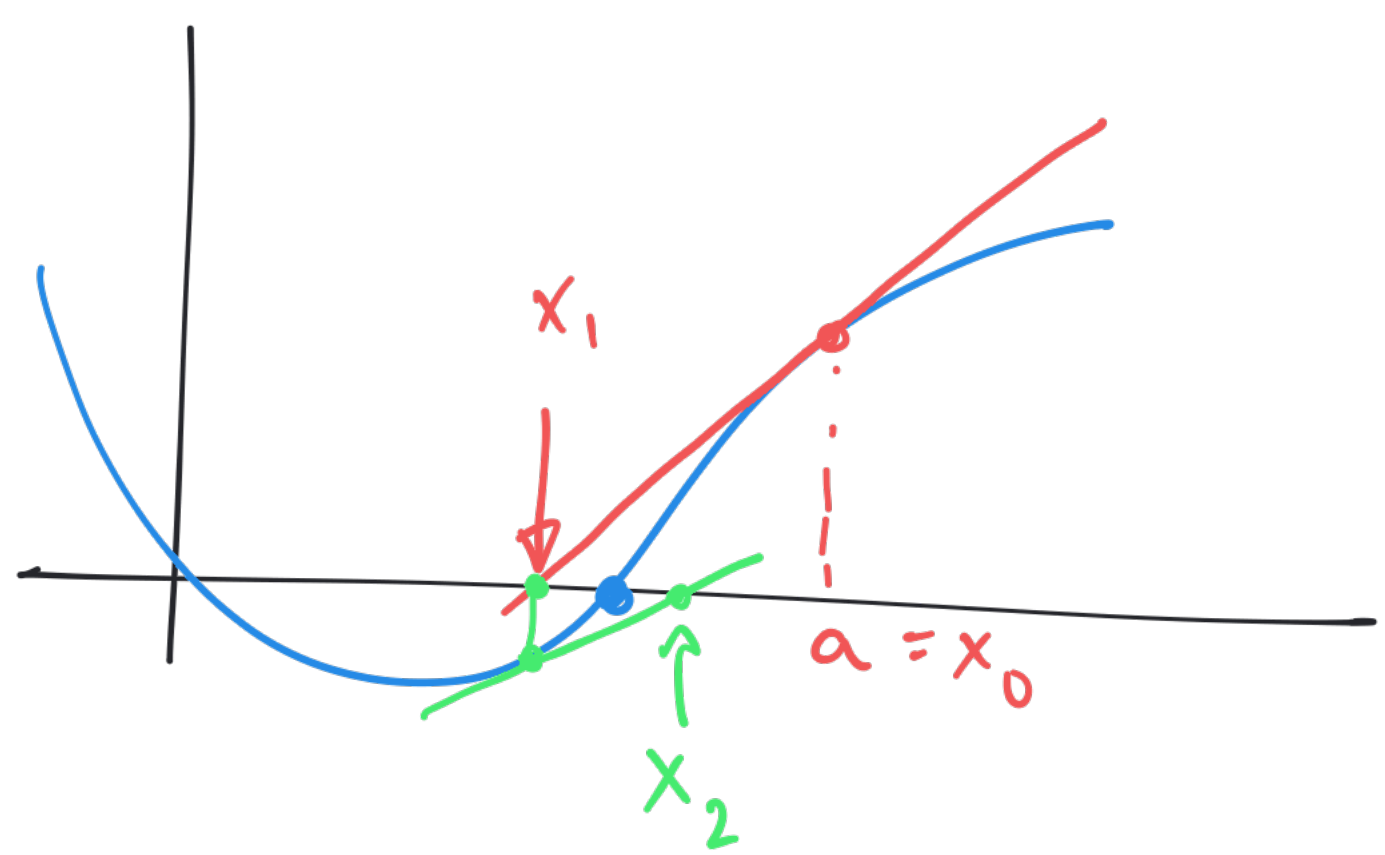
Intro Video: Section 4.8
Newton's Method

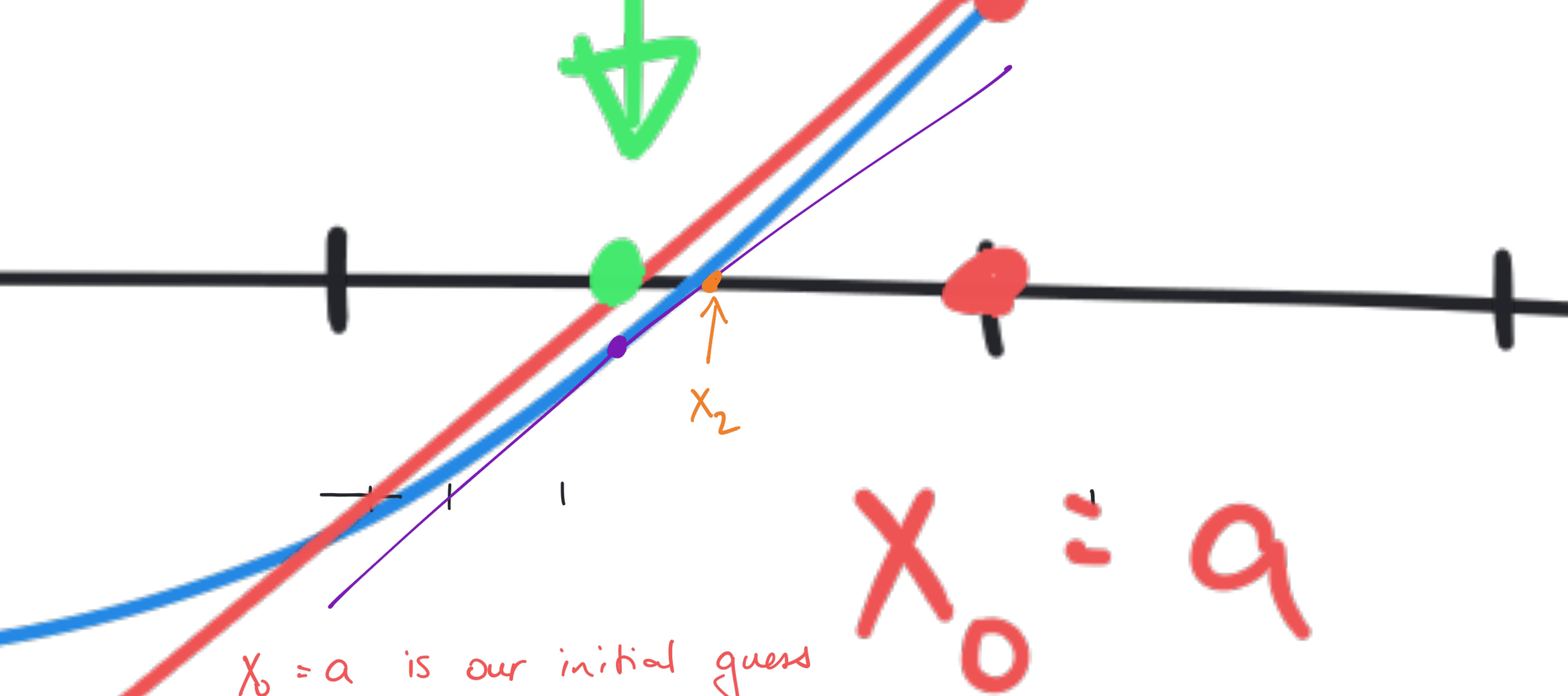
Math F251X: Calculus 1

Question: Given a function $f(x)$, how can we efficiently find a root of $f(x)$?

root: solution to $f(x)=0$

Idea: the tangent line is a pretty good approximation to the function near the point of tangency!





$x_0 = a$ is our initial guess

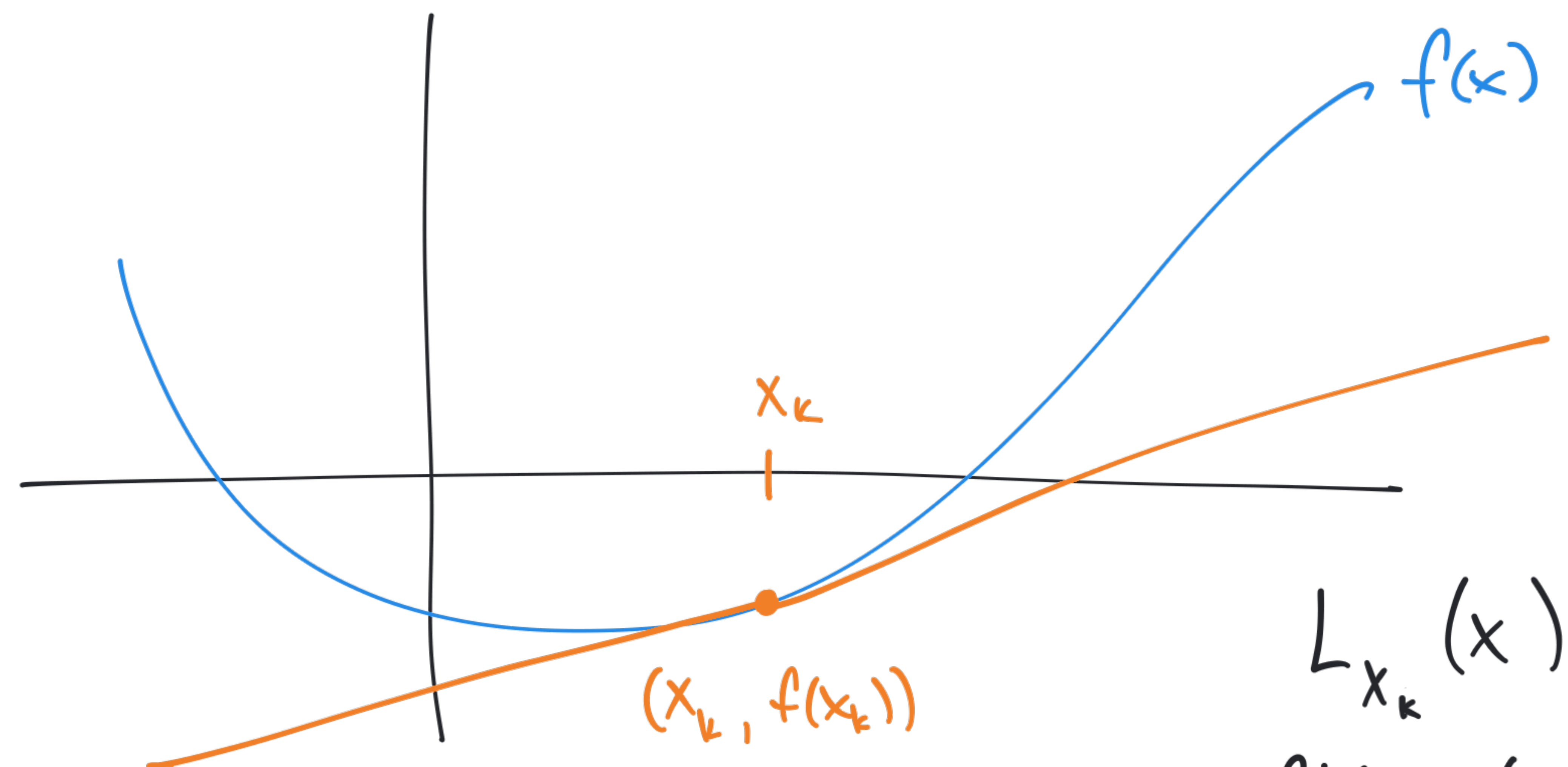
$L_{x_0}(x)$ is the tangent line to $f(x)$ at the point $(x_0, f(x_0))$

→ x_1 is the solution to $\boxed{L_{x_0}(x) = 0}$

$L_{x_1}(x)$ is the tangent line to $f(x)$ at the point $(x_1, f(x_1))$

→ x_2 is the solution to $L_{x_1}(x) = 0$.

Suppose we have some guess x_k



Slope of TL $= f'(x_k)$

Point of tangency $= (x_k, f(x_k))$

$$L_{x_k}(x) = f'(x_k)(x - x_k) + f(x_k)$$

$$L_{x_k}(x) = 0 \Rightarrow$$

$$f'(x_k)(x - x_k) = -f(x_k)$$

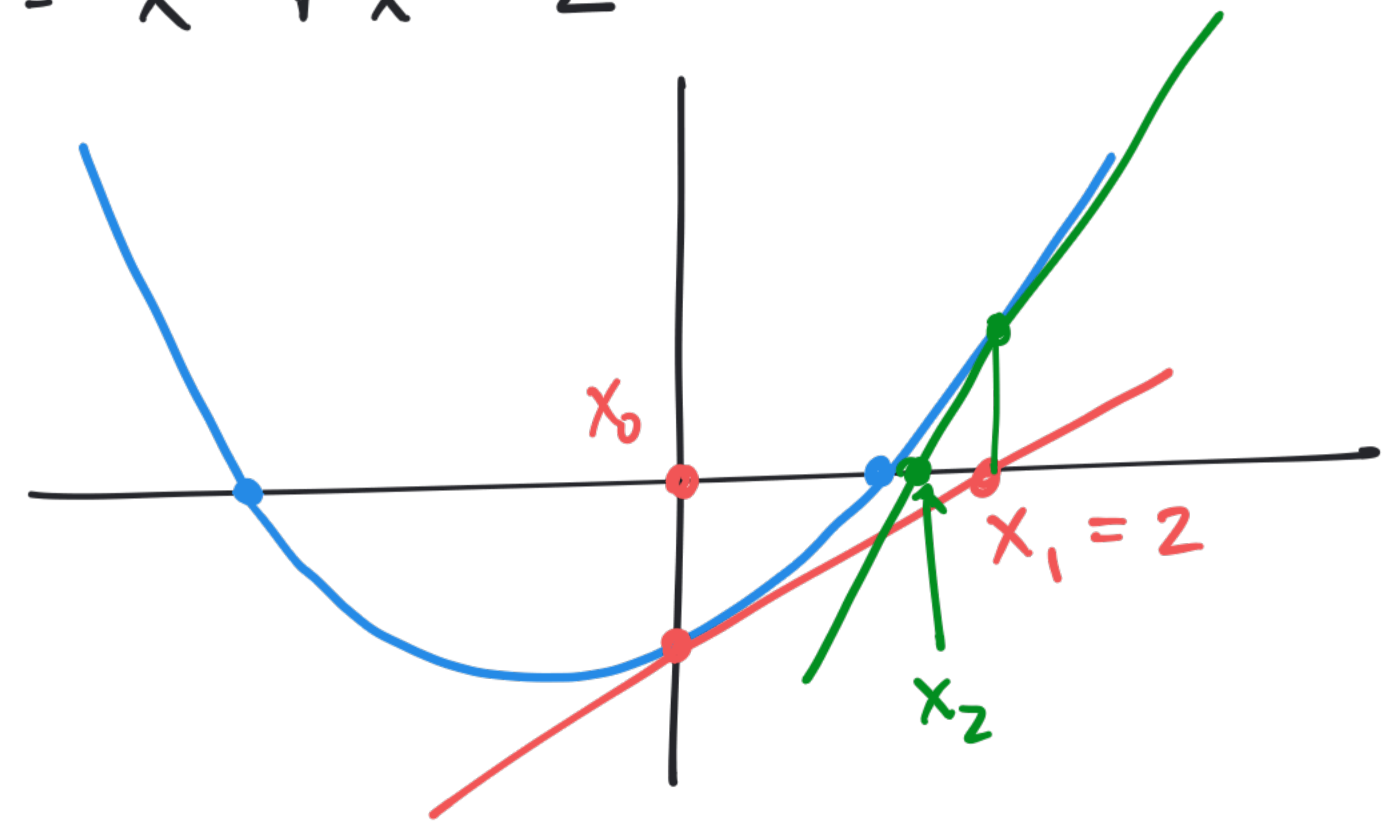
$$x = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

Example:

$$f(x) = (x-1)(x+2) = x^2 + x - 2$$

$f'(x) = 2x + 1$



$x_0 = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0 - \left(\frac{-2}{2(0)+1}\right) = 2$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \left(\frac{4+2-2}{2(2)+1}\right) = 2 - \frac{4}{5} = \frac{6}{5}$$

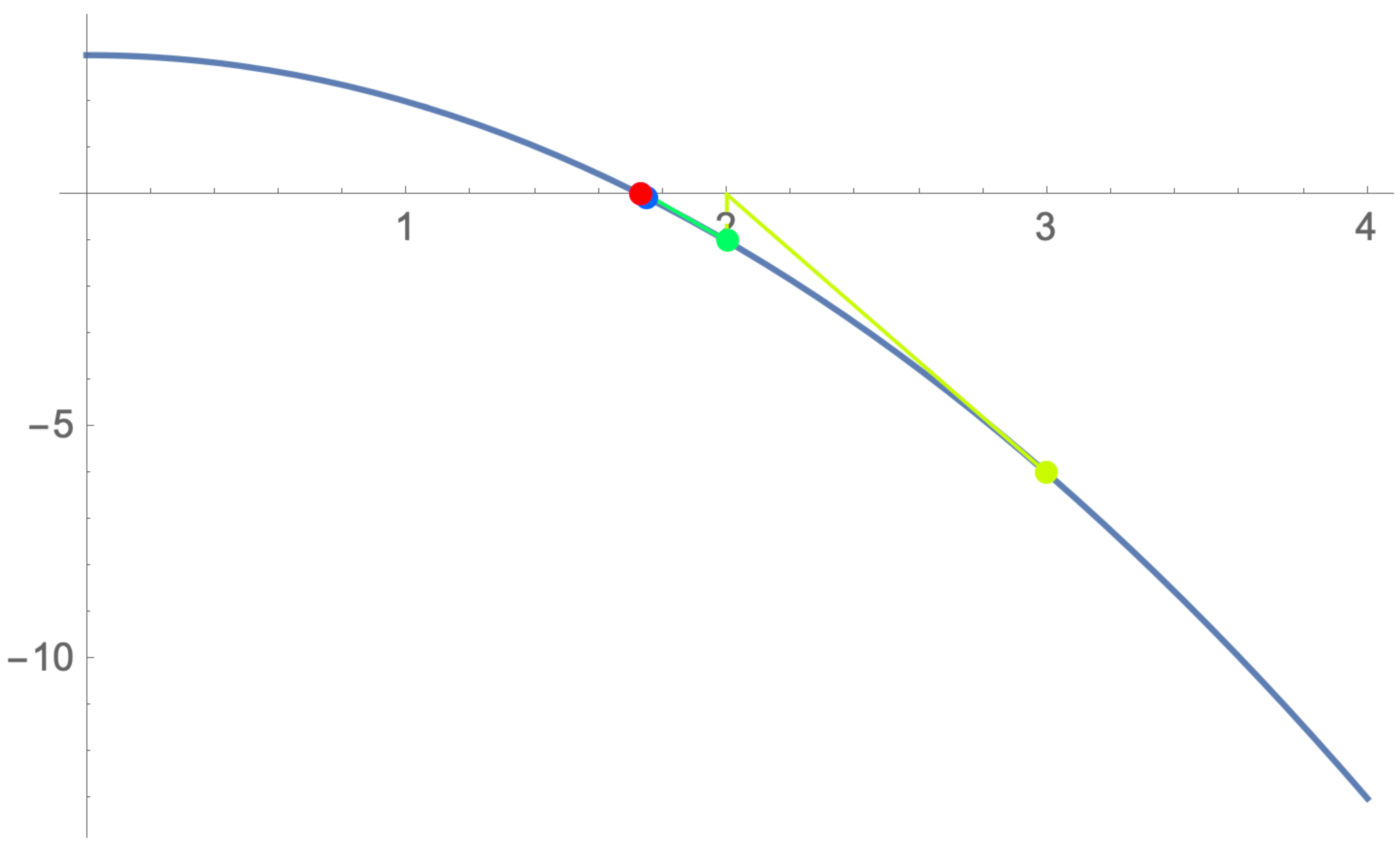
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{6}{5} - \frac{(6/5)^2 + \left(\frac{6}{5}\right) - 2}{2\left(\frac{6}{5}\right) + 1} = \frac{86}{85} \approx 1.011764705$$

$$x_{k+1} = x_k - \frac{(x_k)^2 + x_k - 2}{2x_k + 1} = \frac{x_k(2x_k+1) - (x_k)(x_k{}^2 + x_k - 2)}{2x_k + 1} = \frac{2 + (x_k)^2}{1 + 2x_k}$$

$$x_4 = \frac{21846}{21845} = 1.000045777$$

$$x_5 = \frac{1431655766}{1431655765} = 1.000000000698$$

$$f(x) = 3 - x^2$$



$x_0 = 3.00000000000000$

$x_1 = 2.00000000000000$

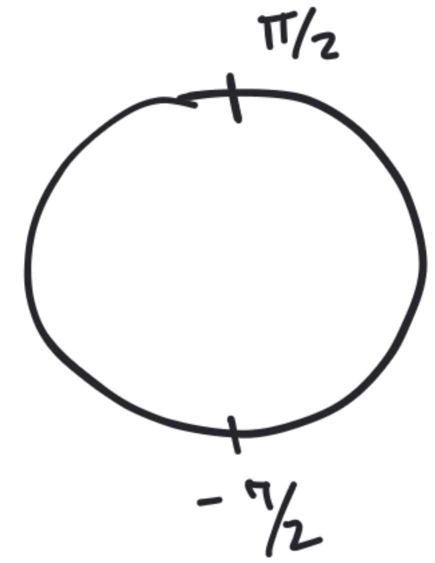
$x_2 = 1.75000000000000$

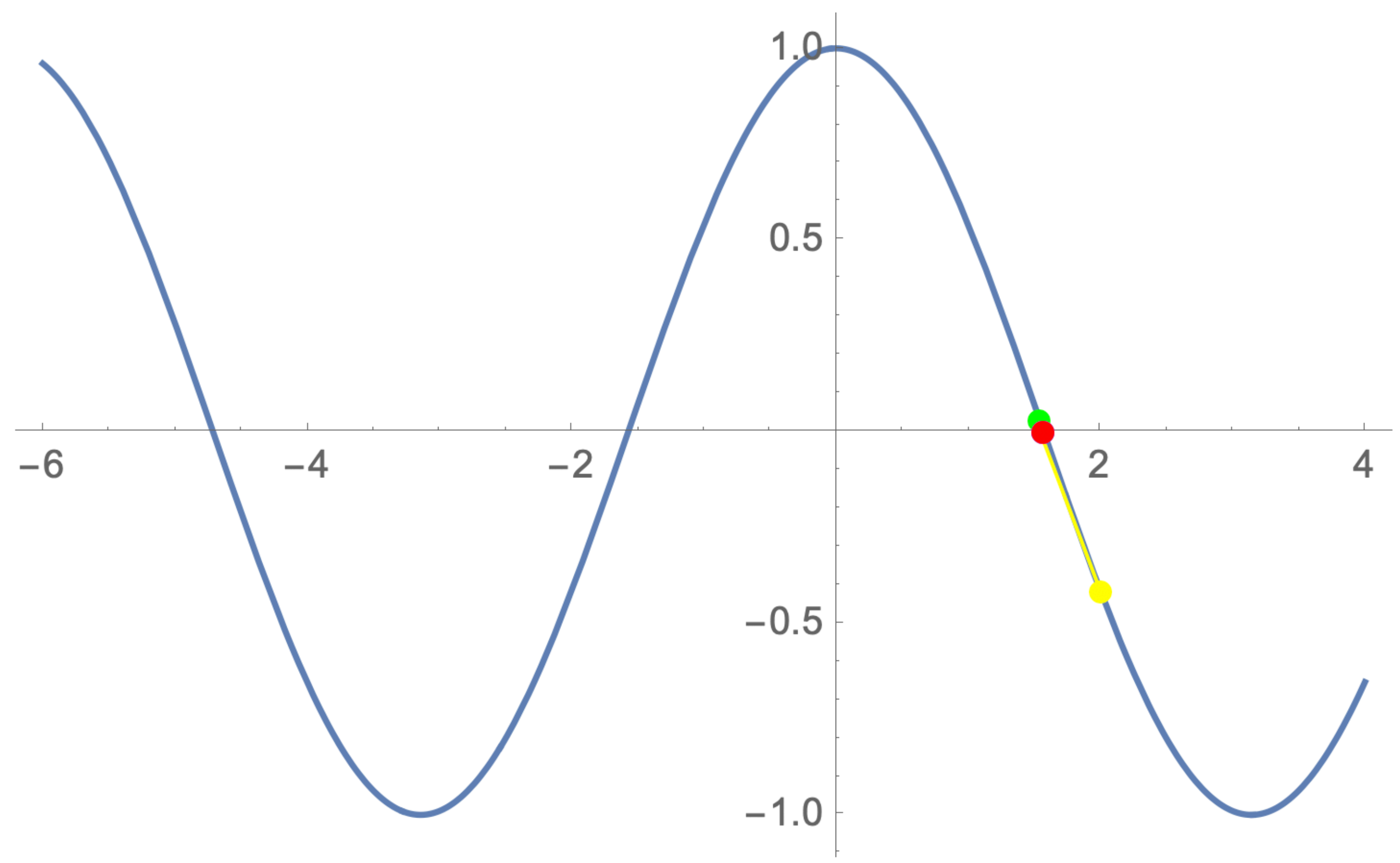
$x_3 = 1.73214285714286$

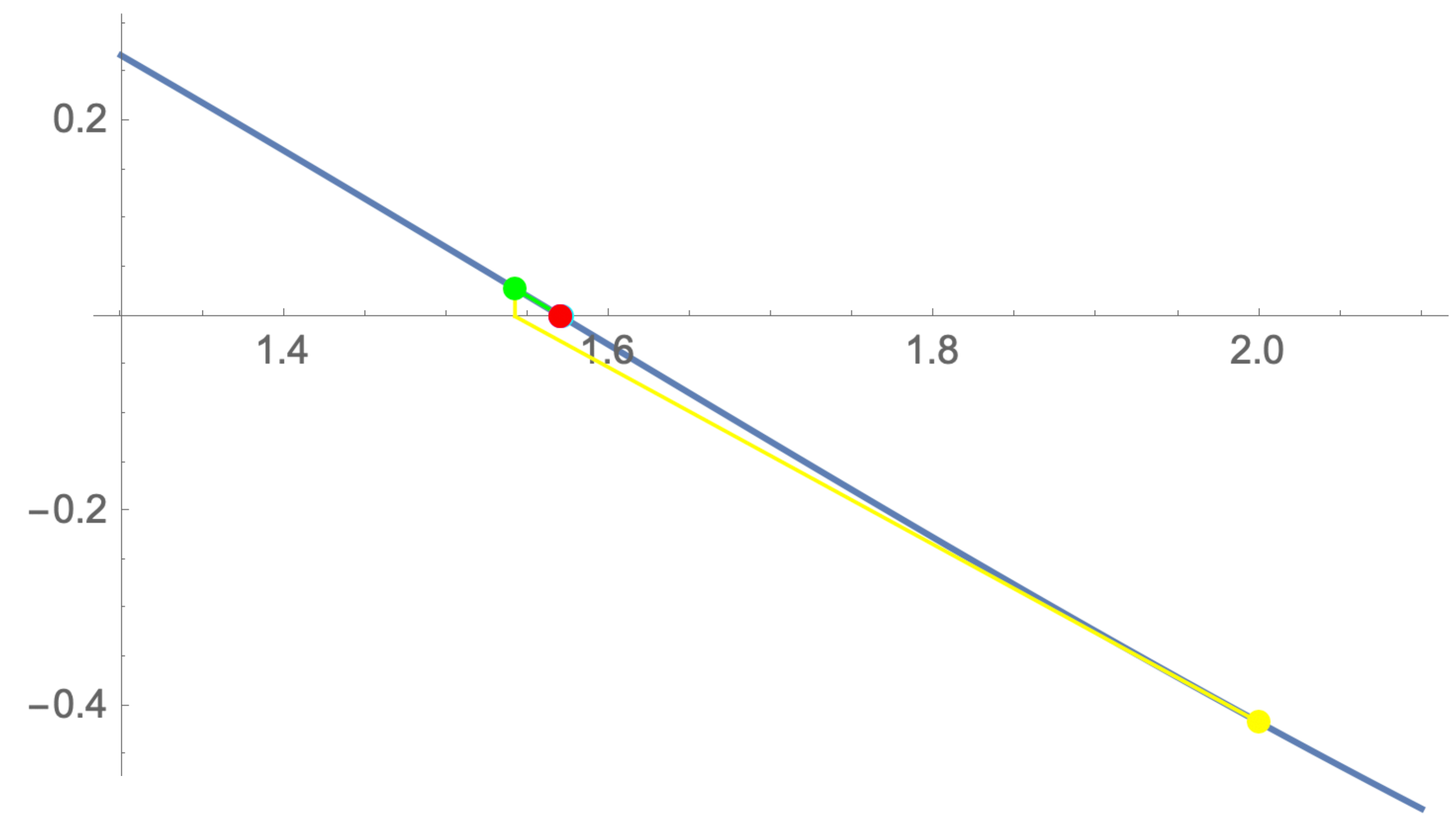
$x_4 = 1.73205081001473$

$x_5 = 1.73205080756888$

$f(x) = \cos(x)$







$x_0 = 2.00000000000000$

$x_1 = 1.54234244563971$

$x_2 = 1.57080400825810$

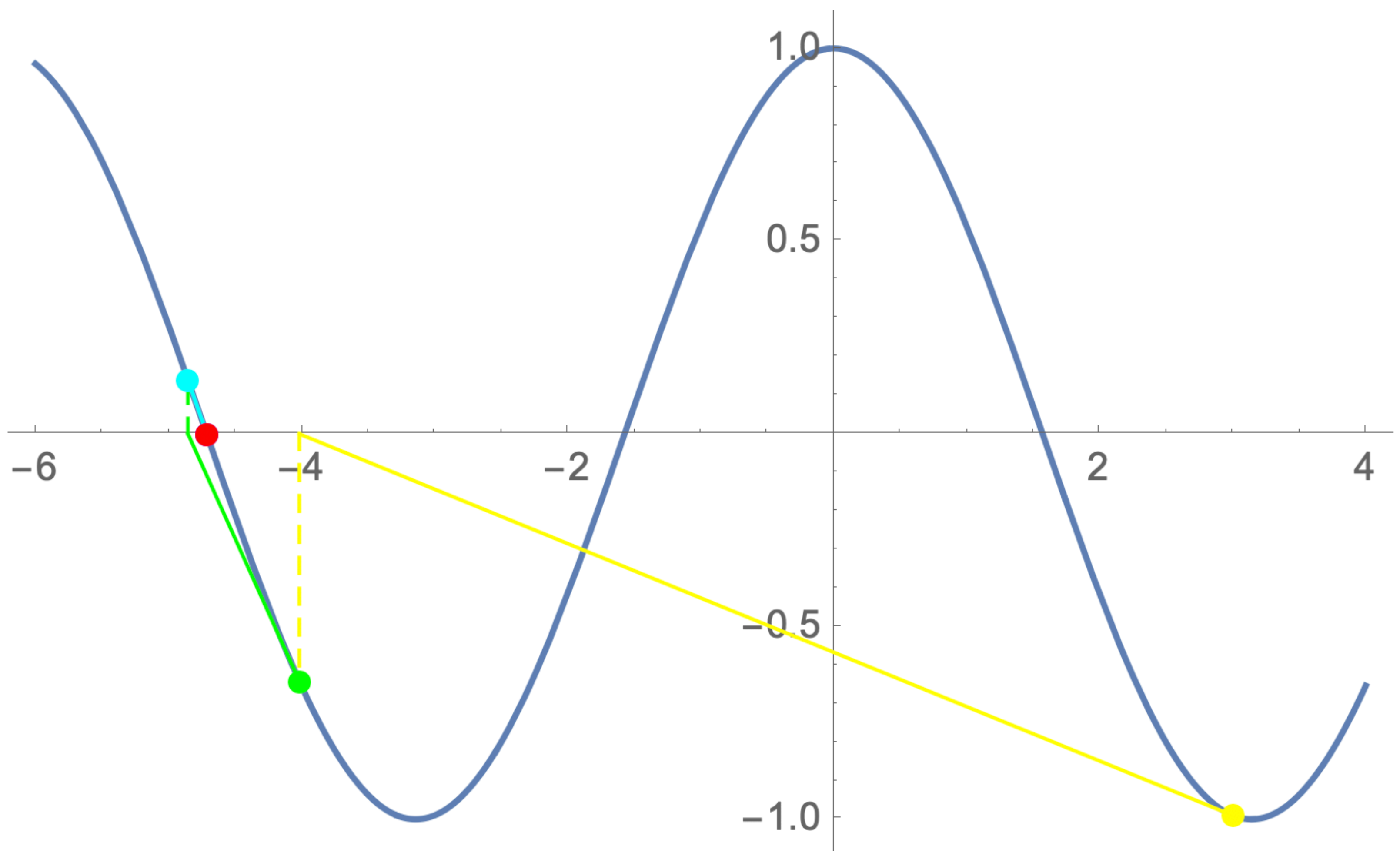
$x_3 = 1.57079632679490$

$x_4 = 1.57079632679490$

$x_5 = 1.57079632679490$

$x_6 = 1.57079632679490$

$f(x) = \cos(x)$, again, but this time let $x_0 = 3$.



$x_0 = 3.0000000000000$

$x_1 = -4.01525255143453$

$x_2 = -4.85265756627868$

$x_3 = -4.71146174116929$

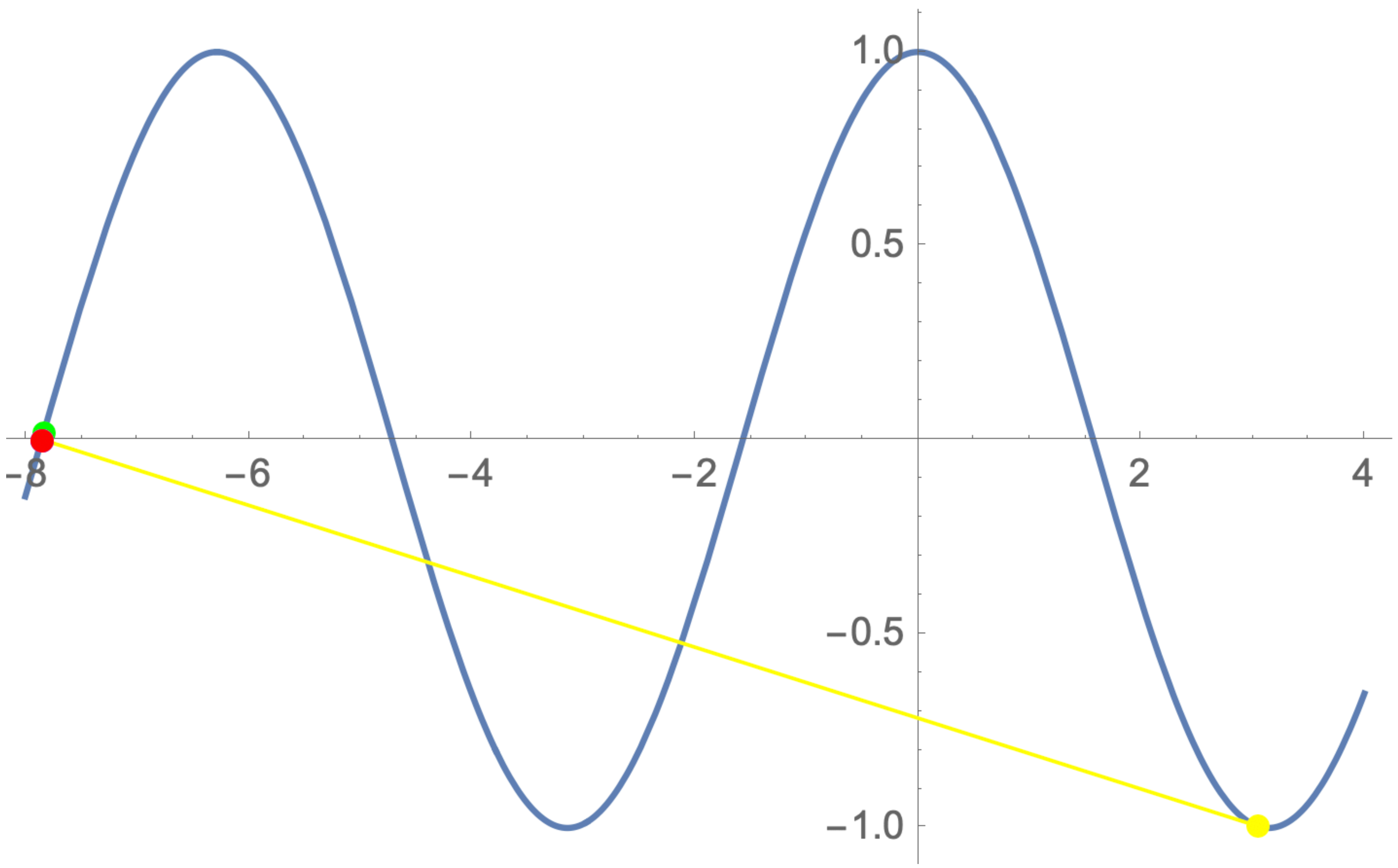
$x_4 = -4.71238898065043$

$x_5 = -4.71238898038469$

$x_6 = -4.71238898038469$

Newton's method may converge on a root that is not what you expected!

$f(x) = \cos(x)$, again, but this time let $x_0 = 3.05$



$x_0 = 3.05$

$x_1 = -7.83736$

$x_2 = -7.85398$

$x_3 = -7.85398$

$x_4 = -7.85398$

$x_5 = -7.85398$

$x_6 = -7.85398$

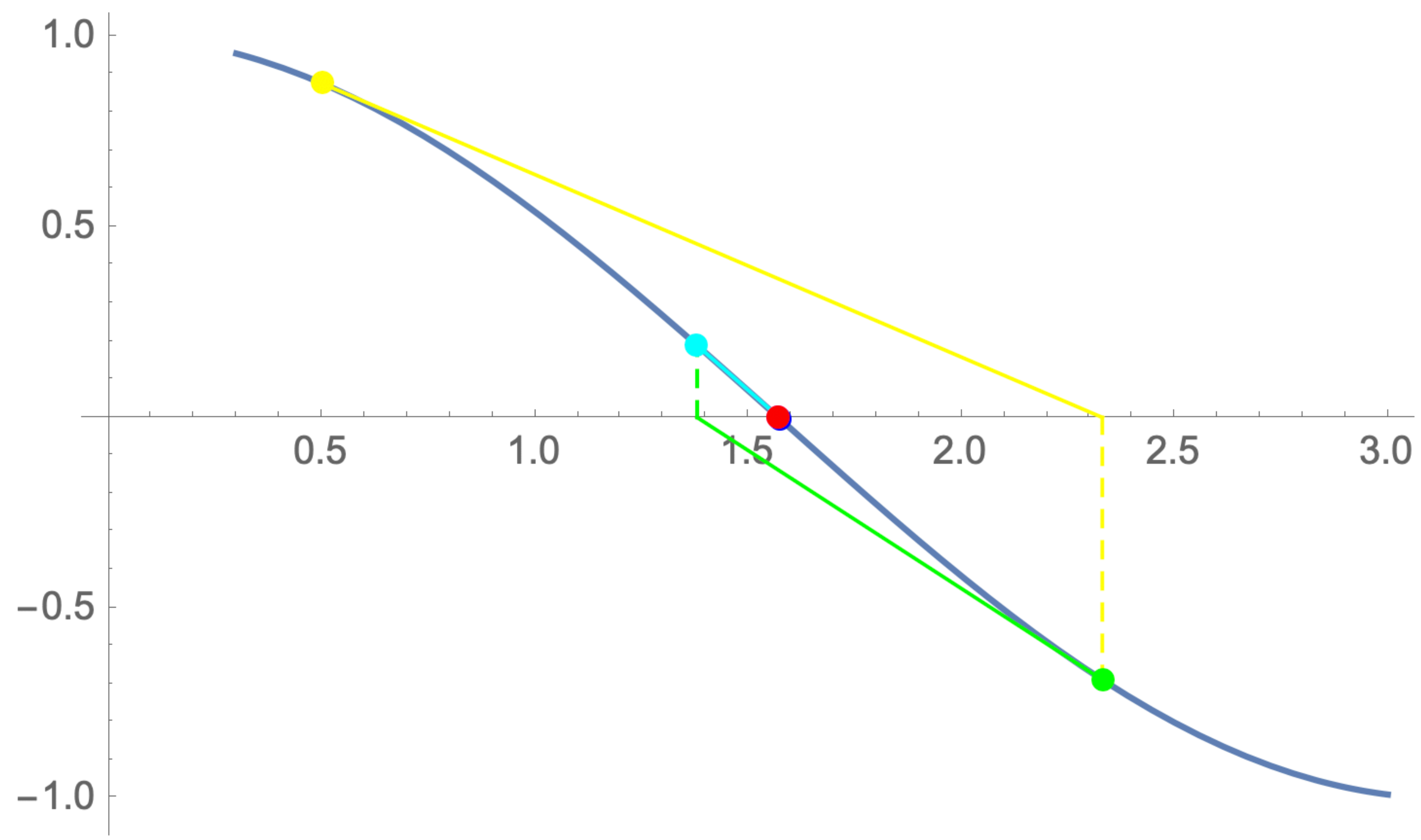
Newton's method may converge on a root that is not what you expected!

$$x_{next} = x_{old} + \frac{f(x_{old})}{f'(x_{old})}$$

Newton's method behaves badly near places where the derivative $= 0$!

Other ways Newton's method may behave unexpectedly...

$f(x) = \cos(x)$



$x_0 = 0.500000000000000$

$x_1 = 2.33048772171245$

$x_2 = 1.38062347483022$

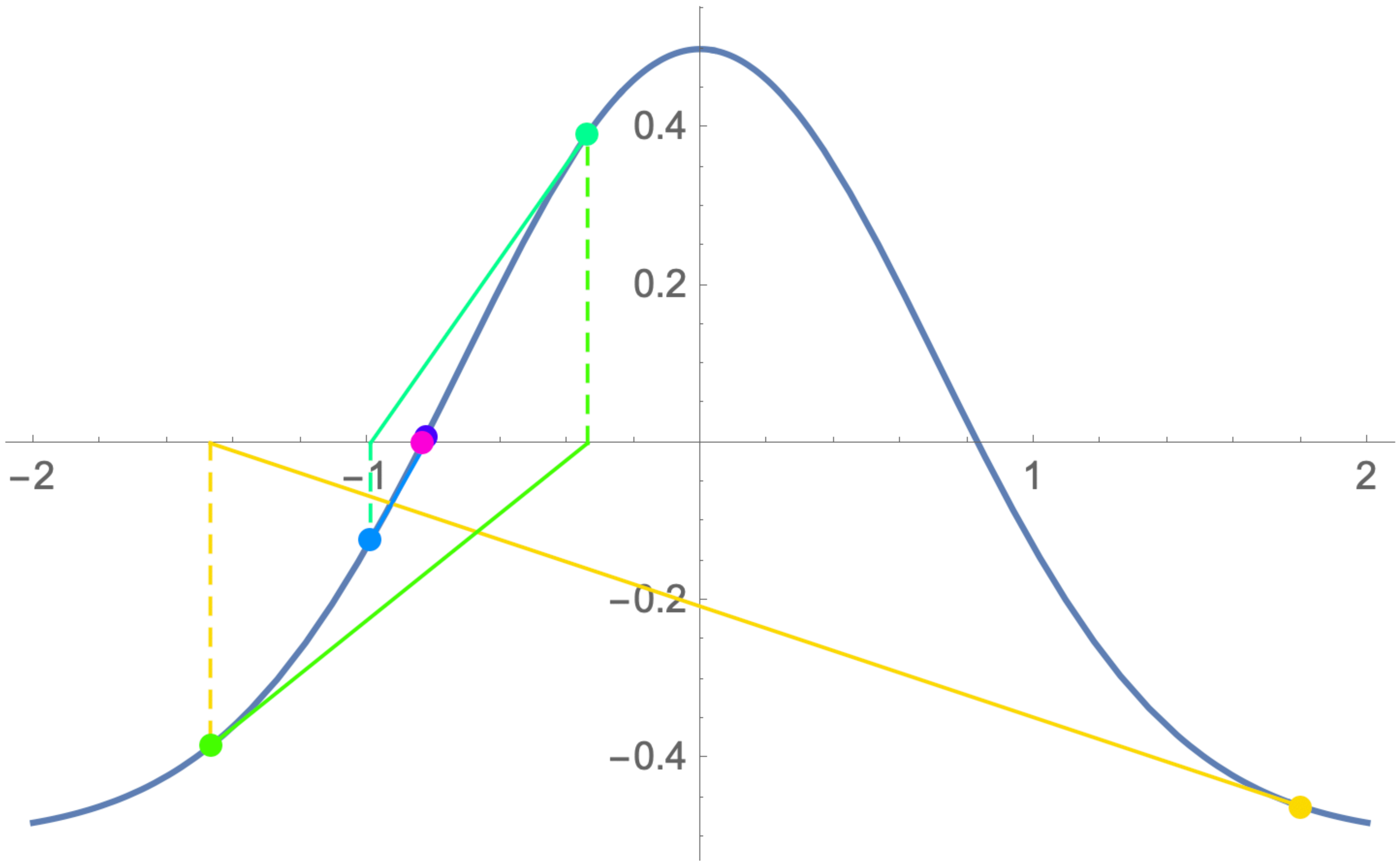
$x_3 = 1.57312256357271$

$x_4 = 1.57079632259884$

$x_5 = 1.57079632679490$

$x_6 = 1.57079632679490$

$$f(x) = e^{-x^2} - \tfrac{1}{2}$$



$x_0 = 1.80000000000000$

$x_1 = -1.46857246490993$

$x_2 = -0.337778076094231$

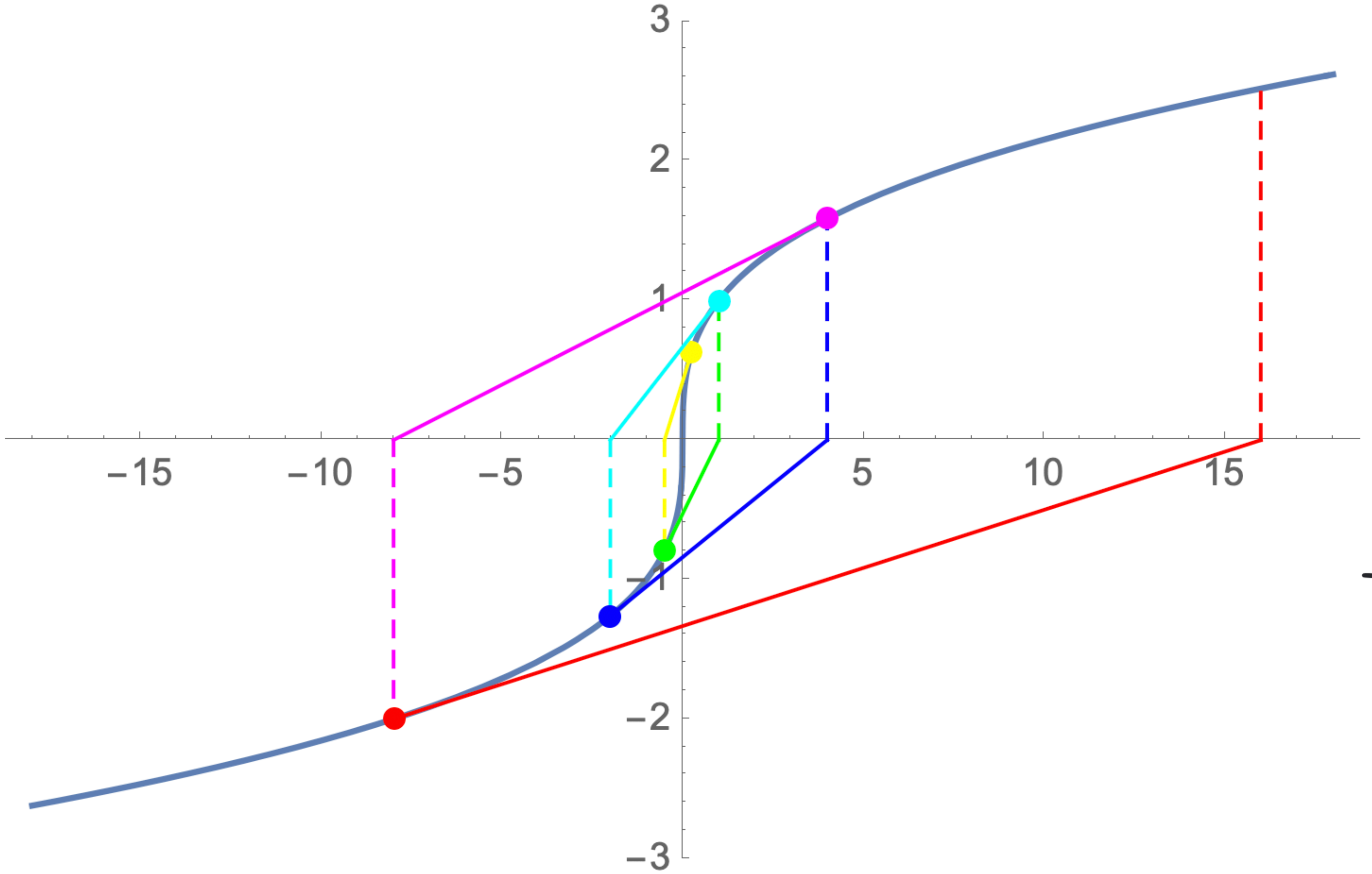
$x_3 = -0.988458618680645$

$x_4 = -0.822389742078299$

$x_5 = -0.832531884642674$

$x_6 = -0.832554611037888$

$f(x) = x^{1/3}$



$x_0 = 0.25000000000000$

$x_1 = -0.50000000000000$

$x_2 = 1.0000000000000$

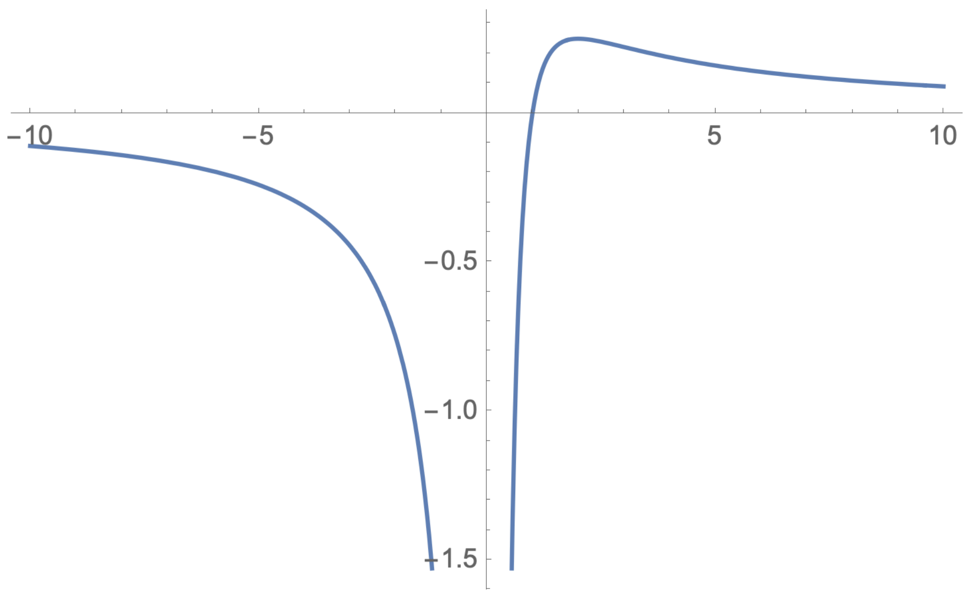
$x_3 = -2.0000000000000$

$x_4 = 4.0000000000000$

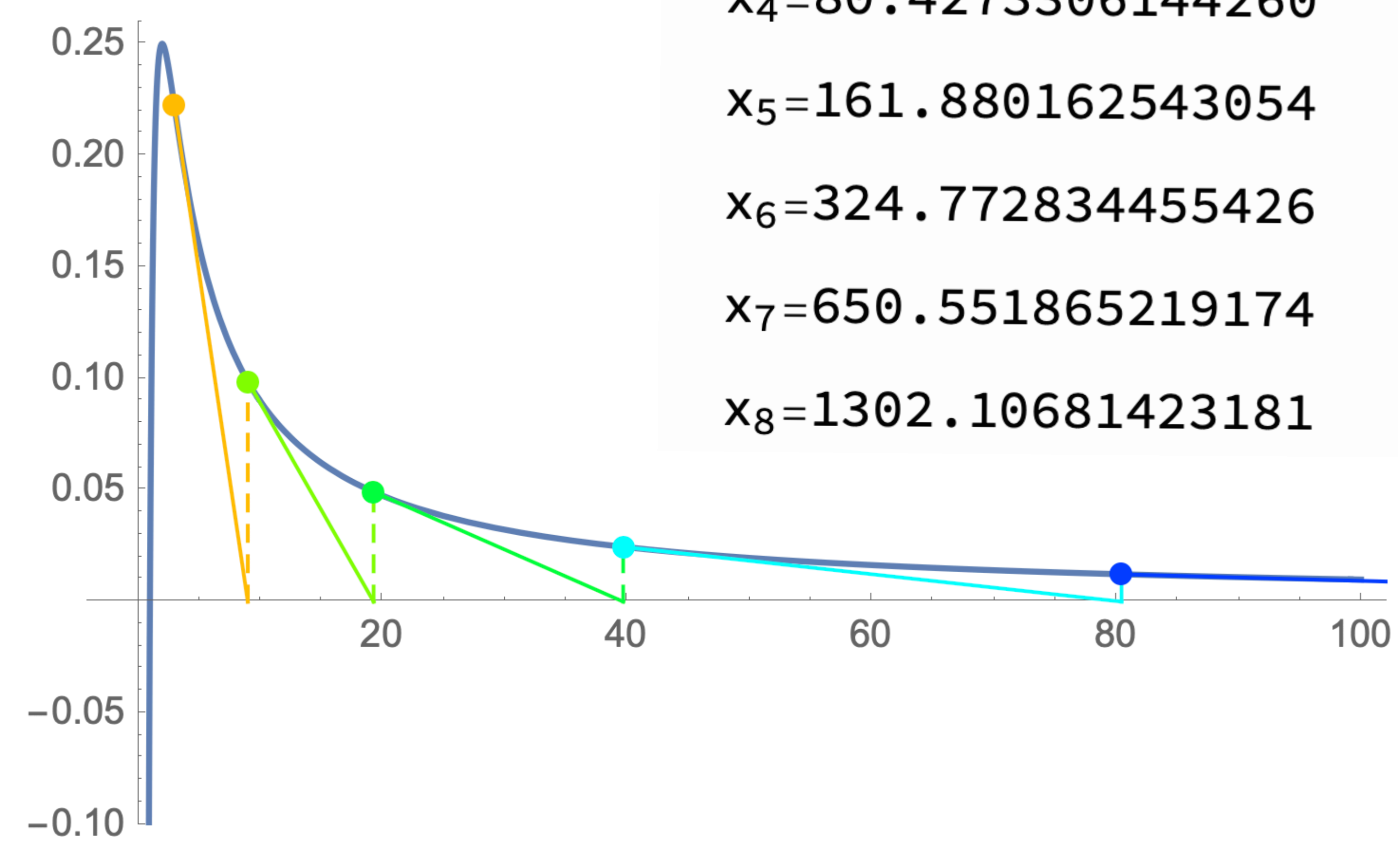
$\boxed{x_5 = -8.0000000000000}$

$\boxed{x_6 = 16.000000000000}$

$$f(x) = \frac{x-1}{x^2}$$



$$\left.\begin{aligned} f(1/2) &= -2 \\ f(3) &= = 2/9 \end{aligned}\right\} \text{IVT} \Rightarrow \text{root}$$



x_0=3.00000000000000

x_1=9.00000000000000

x_2=19.2857142857143

x_3=39.6871310507674

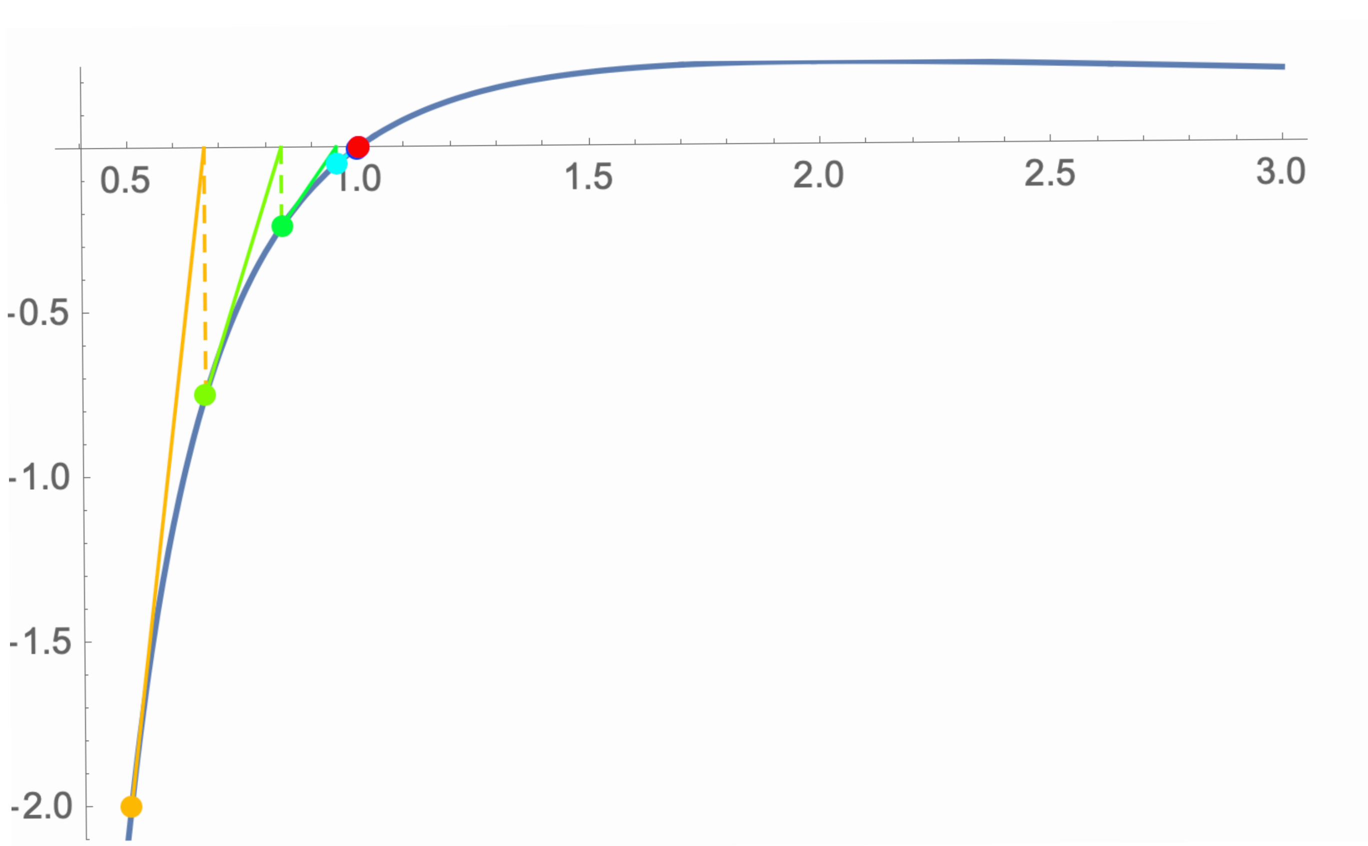
x_4=80.4273306144260

x_5=161.880162543054

x_6=324.772834455426

x_7=650.551865219174

x_8=1302.10681423181



x_0=0.500000000000000

x_1=0.666666666666667

x_2=0.833333333333333

x_3=0.952380952380952

x_4=0.995670995670996

x_5=0.999962680997164

x_6=0.999999997214688

x_7=1.00000000000000

x_8=1.00000000000000